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Nonlinear Inequalities and Entropy-Concurrence Plane

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Abstract Nonlinear inequalities based on the quadratic Renyi entropy for mixed two-qubit states are characterized on the Entropy-Concurrence plane. This class of inequalities is stronger than Clauser-Horne-Shimony-Holt (CHSH) inequalities and, in particular, is violated "in toto" by the set of Type I Maximally-Entangled-Mixture States (MEMS I).

Keywords Entanglement · Entropic inequalities

Introduction

Entanglement, "*the* characteristic trait of quantum mechanics" [1], has been identified as a fundamental physical resource for quantum computation and information, and its quantification and detection have been the subject of considerable research. However, despite a remarkable progress in the field, the so-called *separability problem*, the question whether a state ϱ is entangled or not, has not a general answer yet.

More precisely, a quantum state described by density matrix ϱ of a system composed of two subsystems of dimension N and M, respectively, is called entangled [2] iff it cannot be written as a separable state of the form

$$\sigma = \sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| \otimes \left|\phi_k\right\rangle \left\langle\phi_k\right|, \tag{1}$$

where $p_k \geq 0$ and $\sum p_k = 1$.

Currently the most important criterion for deciding whether a given state is entangled or not is related to the semidefinite positivity of the partial transpose ϱ^{T_A}: separable states have a positive semidefinite partial transpose PPT, hence all non-PPT states are entangled. For systems with 2×2 and 2×3 dimensional Hilbert spaces the PPT-criterion also turns out to be sufficient [3], but for higher dimensional systems there exist PPT entangled states.

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Further, a complete characterization of separable states exists based on "*entanglement witness*". Briefly speaking, entanglement witnesses are operators that are designed directly for distinguishing between separable and entangled states [3–5]. A Hermitian operator W is called an entanglement witness if it has a positive expectation value with respect to all separable states, $\mathrm{Tr}(W\sigma) \geq 0$. The negative expectation value is hence a signature of entanglement, and a state with $\mathrm{Tr}(W\varrho) < 0$ is said to be detected by the witness. The latter condition offers the possibility of experimental detection of entanglement via the measurement of W, an observable which "*witnesses*" the quantum correlations in ϱ.

Historically, a violation of Bell's inequalities [6–10] provided the first test for entanglement. Bell's inequalities were originally designed to prove that quantum mechanics is incompatible with Einstein, Podolsky and Rosen (EPR) local realistic view of the world [11] but, within quantum mechanics, they can be also regarded as non-optimal linear witness operators.

Geometrically, separable states form a convex set in the space of all density matrices of a given system and one might expect that special types of nonlinear witnesses can approximate the convex set of the separable states better than linear ones. In particular, following Schroedinger remarks on relations between the information content of the total system and its subsystems, some separability criteria in terms of entropic uncertainty relations were derived.

Classically, if a system is formed by different subsystems, complete knowledge of the whole system implies that the sum of the information of the subsystems makes up the complete information for the whole system. The Shannon entropy $H(X)$ of a single random variable is never larger than the Shannon entropy of two random variables, that is: $H(X,Y) \geq H(X), H(Y)$. In the quantum world, there exist states of composite systems for which we might have the complete information, while our knowledge about the subsystems might be very poor or null. The canonical example is given by a pair of qubits A and B prepared in the maximally entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$. The von Neumann entropy $S(A)$ of qubit A is equal to 1, compared with a von Neumann entropy $S(A,B)$ of 0 for the joint system. It has been shown [12–14] that for separable states the relation

$$S(A,B) \geq S(A), S(B), \tag{2}$$

holds as a consequence of its concavity [15] but, unfortunately, the inequalities (2) are not sufficient to characterize separability.

The idea to use higher order (nonlinear) entropic inequalities as separability-vs-entanglement criteria for mixed states born when Cerf and Adami [16] and the Horodecki family [3, 17] recognized that conditional Rènyi entropies are non-negative for separable states and it was recently proposed by several groups [18–23], in the form of conditional Tsallis entropies. These entropic inequalities are satisfied by all separable states and are known to be stronger than all Bell-CHSH inequalities.

Recently Derkacz and Jakóbczyk [24, 25] studied the relationship between entanglement, as measured by concurrence $C(\varrho)$, mixedness, as measured by linear entropy $S_L(\varrho)$, and Bell-CHSH violation. These authors showed that the subset Λ on the (C, S_L) plane, previously investigate by Munro et al. [26], is the sum of disjoint subsets Λ_V, Λ_{NV} and Λ_0, with the following properties: states belonging to Λ_V violate CHSH inequalities, states belonging to Λ_{NV} satisfy CHSH inequalities, states from Λ_0, different but with the same entropy and concurrence, can violate or satisfy CHSH inequalities.

Following Derkacz and Jakóbczyk, in this paper the relationship between two-qubit states entanglement and the violation of entropic inequalities on the (C, S_L) plane is investigated.

Non-linear Entropies

The quantum Rėnyi entropy depending on the entropic parameter $\alpha \in \mathbb{R}$ is given by

$$S_\alpha(\varrho) = \frac{\log \mathrm{Tr}(\varrho^\alpha)}{1-\alpha}, \tag{3}$$

where S_0, S_1, S_∞ reduce to the logarithm of the rank, the von Neumann entropy and the negative logarithm of the operator norm, respectively. The conditional Rėnyi entropy reads

$$S_\alpha(B|A;\varrho) := S_\alpha(\varrho) - S_\alpha(\varrho_A). \tag{4}$$

The Tsallis entropy, given by

$$T_\alpha(\varrho) := \frac{1-\mathrm{Tr}(\varrho^\alpha)}{1-\alpha}, \tag{5}$$

is non-negative, concave (convex) for $\alpha > 0$ ($\alpha < 0$) and reduces the von Neumann entropy in the limit $\alpha \to 1$. The conditional Tsallis entropy reads

$$T_\alpha(B|A;\varrho) = \frac{\mathrm{Tr}(\varrho_A^\alpha) - \mathrm{Tr}(\varrho^\alpha)}{(1-\alpha)\,\mathrm{Tr}(\varrho_A^\alpha)}. \tag{6}$$

Concerning positivity, however, the two conditional entropies are equivalent, i.e.:

$$T_\alpha(B|A;\varrho) \geq 0 \quad \text{and} \quad S_\alpha(B|A;\varrho) \geq 0, \tag{7}$$

which is equivalent to

$$\begin{aligned} \mathrm{Tr}\left(\varrho_A^\alpha\right) &\geq \mathrm{Tr}(\varrho^\alpha) \quad \text{for } \alpha > 1, \\ \mathrm{Tr}\left(\varrho_A^\alpha\right) &\leq \mathrm{Tr}(\varrho^\alpha) \quad \text{for } 0 \leq \alpha < 1. \end{aligned} \tag{8}$$

The conditional Tsallis/Rėnyi entropies, involving higher power ($\alpha > 1$) of density matrix ϱ, provide a more stringent criterion for separability [27].

Entropic Inequalities and Entropy-Concurrence Plane

The aim of this section is to obtain the subset of the entanglement-mixedness plane corresponding to violation of quadratic entropic inequalities ($\alpha = 2$). In this case it is possible to extract a nonlocal and nonlinear quantity, namely, the Renyi entropy, from local measurements on two pairs of polarization-entangled photons as showed in [28].

The entanglement can be quantified by the quantity $C(\varrho)$ which is known in literature as *concurrence*. Wootters has derived an analytic formula for the concurrence of two-qubit states [29]:

$$C(\varrho) = 2\max\{\lambda_j\} - \sum_j \lambda_j, \tag{9}$$

where λ_j are the square roots of eigenvalues of the matrix $\tilde{\varrho} = \varrho(\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y)$ and ϱ^* denotes the complex conjugate of density operator ϱ. The mixedness measure is the so-called *linear entropy* and is based on the purity of a state $P = \mathrm{Tr}(\varrho^2)$. The linear entropy S_L

for $C^2 \otimes C^2$ systems is defined via

$$S_L = \frac{4}{3}\left[1 - \mathrm{Tr}\left(\varrho^2\right)\right], \tag{10}$$

and ranges from 0 to 1 (for a maximally mixed state).

In the entanglement-mixedness or, in this case, concurrence-entropy plane, we can start considering the class the class $\mathcal{E}_0$ of states

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & \frac{1}{2}ce^{i\vartheta} & 0 \\ 0 & \frac{1}{2}ce^{-i\vartheta} & b & 0 \\ 0 & 0 & 0 & 1-a-b \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} &c \in [0,1], \quad a, b \geq 0, \quad \vartheta \in [0, 2\pi], \\ &ab \geq \frac{c^2}{4} \quad \text{and} \quad a + b \leq 1, \end{aligned} \tag{12}$$

from the positive definiteness of ϱ. For the class of states $\mathcal{E}_0$, the normalized linear entropy reads

$$S_L = \frac{4}{3}\left(1 - a^2 - b^2 - (1-(a+b))^2 - \frac{c^2}{2}\right), \tag{13}$$

and the concurrence is given by

$$C(\varrho) = c. \tag{14}$$

The boundary value of (13) for fixed c, a and b, such that conditions (12) are satisfied, is given by

$$S_{L\max 1}(c) = \frac{8}{3}c(1-c), \tag{15}$$

for $c \in [\frac{2}{3}, 1]$ and $a = b = \frac{c}{2}$,

$$S_{L\max 2}(c) = \frac{8}{9} - \frac{2}{3}c^2, \tag{16}$$

for $c \in (0, \frac{2}{3})$ and $a = b = \frac{1}{3}$. Then $S_{L\max}(c)$ is reached by the so-called *Maximally Entangled Mixed States* (MEMS's). These are the states which maximize the entanglement degree for a given value of the linear entropy (purity). In particular we can distinguish two families of these states, I and II, defined as

$$\varrho_1(c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c}{2} & \frac{1}{2}ce^{i\vartheta} & 0 \\ 0 & \frac{1}{2}ce^{-i\vartheta} & \frac{c}{2} & 0 \\ 0 & 0 & 0 & 1-c \end{pmatrix}, \quad \text{MEMS I}, \tag{17}$$

$$\varrho_2(c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2}ce^{i\vartheta} & 0 \\ 0 & \frac{1}{2}ce^{-i\vartheta} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \text{MEMS II}. \tag{18}$$

Now let us consider the structure of the set $\Lambda_{\mathcal{E}_0}$ defined by the frontiers (15) and (16).

Theorem 1 *Entropic inequalities disjoin the set* $\Lambda_{\mathcal{E}_0}$ *in a sum subsets* Λ_{V_E}, Λ_{0_E} *and* Λ_{NV_E}:

1. If $(s,c) \in \Lambda_{V\_E}$, then every state $\varrho \in \mathcal{E}_0$ such that $S_L(\varrho) = s$ and $C(\varrho) = c$ violates entropic inequalities.
2. If $(s,c) \in \Lambda_{0\_E}$, then there exist states $\varrho_1, \varrho_2 \in \mathcal{E}_0$ such that $S_L(\varrho_1) = S_L(\varrho_2) = s$ and $C(\varrho_1) = C(\varrho_2) = c$, but ϱ_1 violates entropic inequalities, while ϱ_2 does not violate entropic inequalities.
3. If $(s,c) \in \Lambda_{NV\_E}$, then every state $\varrho \in \mathcal{E}_0$ such that $S_L(\varrho) = s$ and $C(\varrho) = c$ does not violate entropic inequalities.

Proof Following Derkacz and Jakóbczyk let us introduce the new variables

$$x = \frac{1}{\sqrt{2}}(a-b), \qquad y = \frac{1}{\sqrt{2}}\left(a+b-\frac{2}{3}\right). \tag{19}$$

Each state $\varrho \in \mathcal{E}_0$ is now defined by the point $(x,y) \in X_+$ where

$$X_+ = \left\{(x,y) : y^2 + \frac{2\sqrt{2}}{3}y - x^2 - \frac{c^2}{2} + \frac{2}{9} \geq 0,\ y \leq \frac{1}{3\sqrt{2}}\right\}, \tag{20}$$

and linear entropy $S_L(\varrho)$ is now expressed as

$$S_L(\varrho) = -\frac{8}{3}\left(\frac{x^2}{2} + \frac{3}{2}y^2 + \frac{c^2}{2} - \frac{1}{3}\right). \tag{21}$$

Then the states with the same value $S_L = s$ belong to the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \tag{22}$$

with

$$A = \sqrt{6\left(-\frac{c^2}{12} - \frac{s}{8} + \frac{1}{9}\right)}, \qquad B = \sqrt{2\left(-\frac{c^2}{12} - \frac{s}{8} + \frac{1}{9}\right)}. \tag{23}$$

The entropic inequality $\mathrm{Tr}(\varrho^2) - \mathrm{Tr}(\varrho_A^2) \leq 0$ now reads

$$y^2 - xy + \frac{1}{3\sqrt{2}}y + \frac{1}{3\sqrt{2}}x - \frac{c^2}{4} + \frac{1}{9} \leq 0. \tag{24}$$

For fixed concurrence c the intersection of the level set of the function S_L with X_+ can lie below or above the curve representing the inequality bound (24), or can intersect this line, depending on the value s.

The ellipse can intersect the inequality function (24) for $s \leq \frac{8}{3}c(1-c^2)$ for $c > \frac{1}{2}$ and $s \leq \frac{2}{3}$ for $0 < c \leq \frac{1}{2}$: the part of ellipse above hyperbola $y^2 + \frac{2\sqrt{2}}{3}y - x^2 - \frac{c^2}{2} + \frac{2}{9} = 0$ represents Violating Entropic Inequalities States (VEIS), whereas the remaining part corresponds to states with the same s and c, which are not VEIS (see Fig. 1).

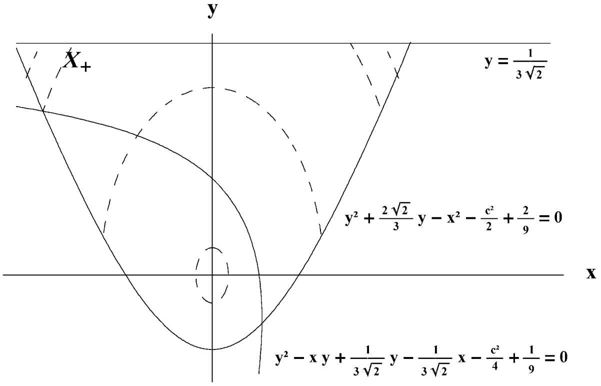

Fig. 1 The figure shows the region of plane $\varrho \in \mathcal{E}_0$ defined by the point $(x, y) \in X_+$ bounded by the *hyperbola* $y^2 + \frac{2\sqrt{2}}{3}y - x^2 - \frac{c^2}{2} + \frac{2}{9} = 0$ and by the *straight line* $y = \frac{1}{3\sqrt{2}}$. For fixed concurrence c the intersection of the level set of the function S_L (represented by *dotted lines*) with X_+ can lie below or above the curve representing the inequality bound (24), or can intersect this line, depending on the value s

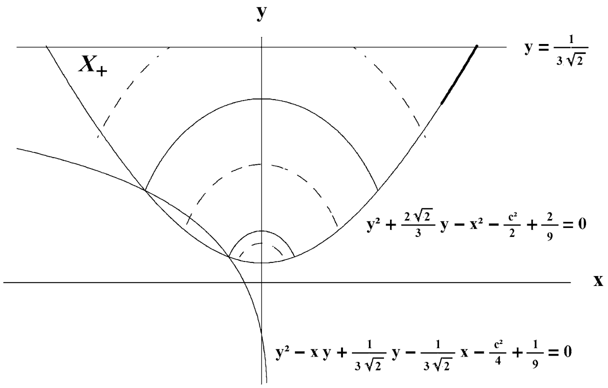

Fig. 2 The curve representing entropic condition intersects the hyperbola only in two points because they have a common asymptote ($y = x - \frac{\sqrt{2}}{3}$). The upper one always lies in the halfplane $x < 0$ and until the intersection of the ellipse representing level set of S_L with the hyperbola is above this point, i.e. for $s < \frac{1}{3}(1 + c^2 - \sqrt{1 - 2c^2})$, all states are VEIS. The lower point for $c > \frac{2}{3}$ lies in the halfplane $x < 0$ and until the intersection of the ellipse representing level set of S_L with the hyperbola is below this point, i.e. $s > \frac{1}{3}(1 + c^2 + \sqrt{1 - 2c^2})$, all states are VEIS

For $s > \frac{2}{3}$ no state violates the entropic inequality. The curve representing entropic condition intersects the hyperbola $y^2 + \frac{2\sqrt{2}}{3}y - x^2 - \frac{c^2}{2} + \frac{2}{9} = 0$ only in two points because they have a common asymptote ($y = x - \frac{\sqrt{2}}{3}$). The upper one always lies in the halfplane $x < 0$ and until the intersection of the ellipse with the hyperbola is above this point, i.e. for $s < \frac{1}{3}(1 + c^2 - \sqrt{1 - 2c^2})$, all states are VEIS. The lower point for $c > \frac{2}{3}$ lies in the halfplane $x < 0$ and until the intersection of the ellipse with the hyperbola is below this point, i.e. $s > \frac{1}{3}(1 + c^2 + \sqrt{1 - 2c^2})$, all states are VEIS.

For $c > \frac{1}{\sqrt{2}}$ the curve representing the inequality bound has no common points with X_+ and all states are VEIS. These conditions define the subsets Λ_{NV_E}, Λ_{V_E} and Λ_{0_E}.

$$\Lambda_{NV\_E} = \left\{(s, c) : 0 < c < \frac{1}{2}, \frac{2}{3} < s \le S_{L2}(c)\right\}$$
$$\cup \left\{(s, c) : \frac{1}{2} \le c < \frac{2}{3}, S_{L1}(c) \le s \le S_{L2}(c)\right\},$$

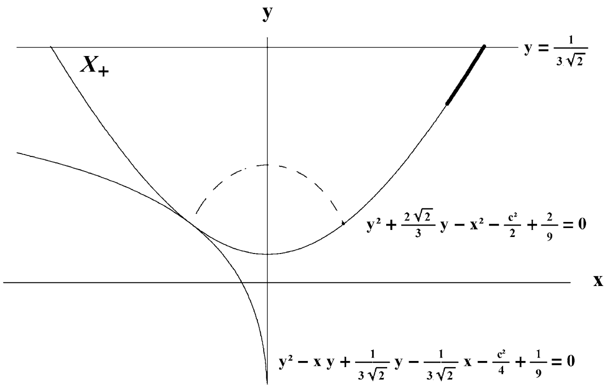

Fig. 3 For $c > \frac{1}{\sqrt{2}}$ the curve representing the inequality bound has no common points with X_+ and all states are VEIS

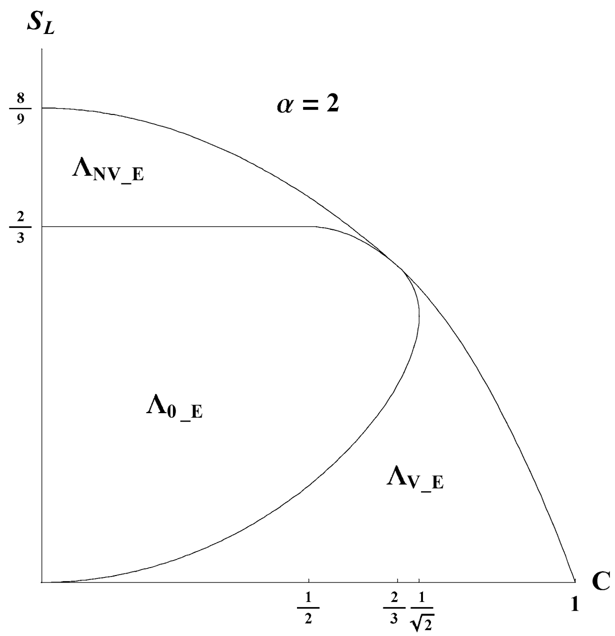

Fig. 4 The figure represents the structure of the set $\Lambda_{\mathcal{E}_0}$ defined by the frontiers (15) and (16). If $(s, c) \in \Lambda_{V\_E}$, then every state $\varrho \in \mathcal{E}_0$ such that $S_L(\varrho) = s$ and $C(\varrho) = c$ violates entropic inequalities. If $(s, c) \in \Lambda_{0\_E}$, then there exist states $\varrho_1, \varrho_2 \in \mathcal{E}_0$ such that $S_L(\varrho_1) = S_L(\varrho_2) = s$ and $C(\varrho_1) = C(\varrho_2) = c$ and ϱ_1 violates entropic inequalities, but ϱ_2 does not violate entropic inequalities. If $(s, c) \in \Lambda_{NV\_E}$, then every state $\varrho \in \mathcal{E}_0$ such that $S_L(\varrho) = s$ and $C(\varrho) = c$ does not violates entropic inequalities

$$
\begin{aligned}
\Lambda_{V\_E} = &\left\{(s, c) : 0 < c < \frac{1}{\sqrt{2}}, 0 \le s < S_{L-}(c)\right\} \\
&\cup \left\{(s, c) : \frac{2}{3} \le c \le \frac{1}{\sqrt{2}}, S_{L+}(c) < s \le S_{L1}(c)\right\} \\
&\cup \left\{(s, c) : c > \frac{1}{\sqrt{2}}, 0 \le s \le S_{L1}(c)\right\}, \\
\Lambda_{0\_E} = &\{(s, c) : \notin \Lambda_V, \Lambda_{NV}\},
\end{aligned}
\tag{25}
$$

where

$$S_{L1}(c) = S_{L\max 1}(c),$$

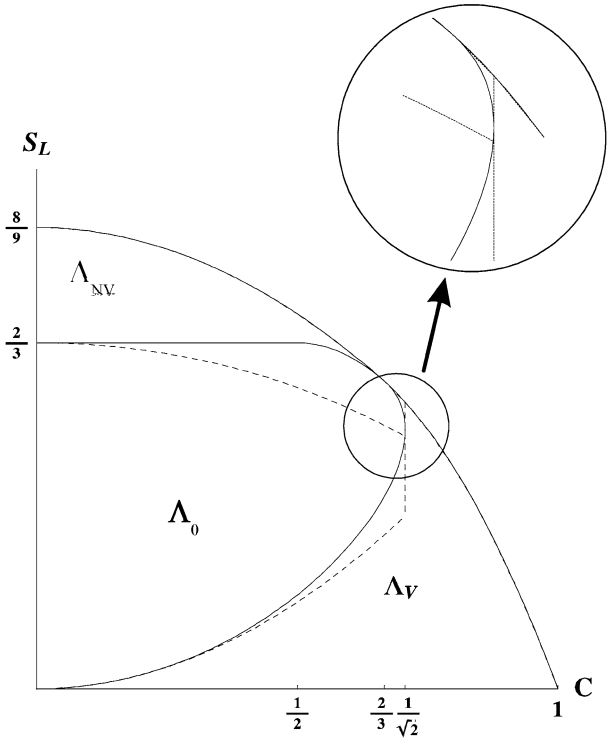

Fig. 5 The figure shows the bounds found from the Nonlinear Inequality (*continuous line*) and CHSH inequality (*dotted line*) on Entropy-Concurrence plane. It is possible to appreciate the larger region where entanglement is detected. The branch of MEMS I for $2/3 \le c \le 1/\sqrt{2}$, violates "*in toto*" the nonlinear inequality, as showed in the inset

$$\begin{aligned} S_{L2}(c) &= S_{L\max 2}(c), \\ S_{L+}(c) &= \frac{1}{3}\left(1 + c^2 + \sqrt{1-2c^2}\right), \\ S_{L-}(c) &= \frac{1}{3}\left(1 + c^2 - \sqrt{1-2c^2}\right). \end{aligned} \tag{26}$$

Figure 5 represents the bounds fixed by the Entropic Inequality in respect to CHSH Inequality. We can see that the region of (C, S_L) plane where entanglement is detected is larger, and there is a reduction of the region where the entanglement is not detected. We can quantify these results in terms of relative area of the different subsets Λ_{V_E}, Λ_{0_E} and Λ_{NV_E} in respect to the total area of the total set Λ corresponding to physical states and we can compare the results with CHSH case:

$$\begin{aligned} &\Lambda_{V\_E} \simeq 28.390\%, &&\Lambda_{V\_CHSH} \simeq 26.577\%, \\ &\Lambda_{0\_E} \simeq 58.155\%, &&\Lambda_{0\_CHSH} \simeq 54.788\%, \\ &\Lambda_{NV\_E} \simeq 13.455\%, &&\Lambda_{NV\_CHSH} \simeq 18.635\%. \end{aligned} \tag{27}$$

It is possible to appreciate the larger region where entanglement is detected by nonlinear inequality in respect to CHSH ones. Moreover CHSH inequality does not detect the branch of MEMS I for $2/3 \le c \le 1/\sqrt{2}$, while nonlinear inequality is violated by these states "in toto" as shown in Fig. 5.

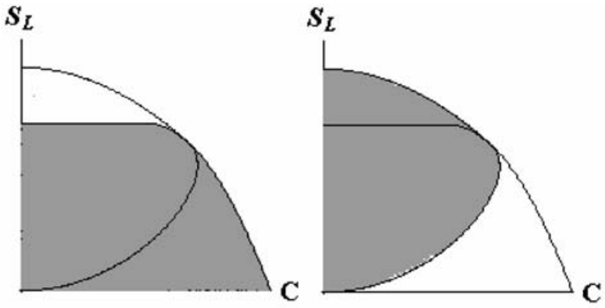

Fig. 6 The numerical analysis of the set $\Lambda_{\mathcal{E}_1}$ shows that its structure is the same of the previous set of density matrices $\Lambda_{\mathcal{E}_0}$, i.e. the bounds of the three regions Λ_{V_E}, Λ_{0_E} and Λ_{NV_E} remain unchanged. The picture on the *left* shows the states which violate the non-linear inequality, the picture on the *right* the states which satisfy the inequality

Let us extend this results to the larger class $\mathcal{E}_1$ of states of the form

$$\varrho = \begin{pmatrix} f & 0 & 0 & \frac{1}{2}de^{i\phi} \\ 0 & a & \frac{1}{2}ce^{i\vartheta} & 0 \\ 0 & \frac{1}{2}ce^{-i\vartheta} & b & 0 \\ \frac{1}{2}de^{-i\phi} & 0 & 0 & 1-a-b-f \end{pmatrix}. \tag{28}$$

For these states the normalized linear entropy reads

$$S_L = \frac{4}{3}\left(1 - a^2 - b^2 - \frac{c^2}{2} - \frac{d^2}{2} - f^2 - (1-a-b-f)^2\right), \tag{29}$$

and the concurrence is given by

$$C(\varrho) = \max(0, C_1, C_2) \tag{30}$$

with

$$\begin{aligned} C_1(\varrho) &= d - \sqrt{ab}, \\ C_2(\varrho) &= c - \sqrt{f(1-a-b-f)}. \end{aligned} \tag{31}$$

The description of the set $\Lambda_{\mathcal{E}_1}$ was made numerically, by generating a very large number of randomly density matrices. The results (see Fig. 6) show that the structure of $\Lambda_{\mathcal{E}_1}$ is the same of the previous set of density matrices $\Lambda_{\mathcal{E}_0}$, i.e. the bounds of the three region Λ_{V_E}, Λ_{0_E} and Λ_{NV_E} remain unchanged. □

Conclusion

In this paper, nonlinear inequalities, based on the quadratic Renyi entropy, were represented on the Entropy-Concurrence plane for the two sets of mixed two-qubit states $\mathcal{E}_0$ and $\mathcal{E}_1$, and a comparison was made with respect to CHSH inequalities. The analysis of higher order ($\alpha > 2$) cases showed other interesting properties of non-linear inequalities and it will be presented successively.

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